

Problem Set III: Comparative static, monotonicity, Cobb-Douglas

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1. MWG 2.F.10: substitution matrix

Consider the following demand function:

$$x_1(p, w) = \frac{p_2}{p_1 + p_2 + p_3} \frac{w}{p_1}$$

$$x_2(p, w) = \frac{p_3}{p_1 + p_2 + p_3} \frac{w}{p_2}$$

$$x_3(p, w) = \frac{p_1}{p_1 + p_2 + p_3} \frac{w}{p_3}$$

1. Compute the substitution matrix. Show that at $p = (1, 1, 1)$ and $w = 1$ it is negative semidefinite but not symmetric.
2. Show that this demand function does not satisfy the weak axiom. [Hint: consider $p = (1, 1, \varepsilon)$ and show that the matrix is not negative semidefinite for $\varepsilon > 0$ small].



Recap: substitution matrix I

Definition (Slutsky substitution matrix)

The *substitution matrix* $S(p, w)$ measures the differential change in the consumption of commodity l due to a differential change in the price of commodity k , when wealth is adjusted so that the consumer can still just afford his original consumption bundle.

The general element of the matrix $S(p, w)$ has the form

$$s_{lk}(p, w) = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w)$$

The matrix in other words tells us the change in the consumption of commodity l solely due to a change in relative prices, when the wealth effect of the price change has been compensated.



Solution: computing the substitution matrix

- We will have to compute the derivative for all couples l, k , and then substitute $p = (1, 1, 1)$ and $w = 1$.
- As an example, the first element of the matrix is given by

$$s_{11}(p, w) = -\frac{p_2}{(p_1 + p_2 + p_3)^2} \frac{w}{p_1} - \frac{p_2}{p_1 + p_2 + p_3} \frac{w}{p_1^2} + \left(\frac{p_2}{(p_1 + p_2 + p_3)p_1} \right) \frac{p_2}{p_1 + p_2 + p_3} \frac{w}{p_1}$$

- Substituting $p = (1, 1, 1)$ and $w = 1$ we get

$$s_{11}(p, w) = -\frac{1}{9} - \frac{1}{3} + \left(\frac{1}{3} \right) \frac{1}{3} = -\frac{1}{3}$$

- repeating the same calculation over all the matrix, we would get

$$s_{lk}(p, w) = \frac{1}{3} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$



Solution: negative semidefiniteness

- The matrix is obviously not symmetric.
- We must then study the definiteness of the matrix, without the usual criteria for symmetric matrices (eigenvalues, k_{th} order principal minors, north-west minors...).
- We will use two results:
 1. If $x(p, w)$ is homogeneous of degree zero and satisfies Walras' law, then $p \cdot S(p, w) = 0$ and $S(w, p) \cdot p = 0$
 2. If M is a square matrix that has the above property and \hat{M} is the matrix obtained by deleting one row and the corresponding column, then M is negative semidefinite iff \hat{M} is negative definite.
- we will hence have to prove that \hat{M} is negative definite.
- This can be done by studying the sign of the quadratic form $z' \hat{M} z$:
$$[v_1, v_2] \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -v_1^2 + v_1 v_2 - v_2^2 = -\left(v_1 - \frac{v_2}{2}\right)^2 - \frac{3}{4}v_2^2$$
- which is always negative. Hence the minor studied is negative definite, and the matrix $S(w, p)$ is negative semidefinite.

Recap: W.A.R.P. in the context of Walrasian demand

Situation	Choice	Implication
(p, w)	$x(p, w)$	$C(B) = x(p, w)$
(p', w')	$x(p', w')$	$C(B) = x(p', w')$

- Given the above two situations, we will apply the logic of the W.A.R.P...
- ...which imposes consistency: if A is chosen when B is available, then in another choice when B is chosen, A must not be available.

Hypothesis	Choice	Implication
if $p \cdot x(p', w') \leq w$	$x(p, w)$	In situation (p, w) , $x(p', w')$ is available, but <i>not</i>
if $x(p', w') \neq x(p, w)$	-	The two demanded bundles are not identical
then $p' \cdot x(p, w) > w'$	$x(p', w')$	$x(p, w)$ must be unaffordable

- Since the agent reveals a preference for $x(p, w)$ when $x(p', w')$ is affordable, we require him to stick to this preference when both of them are affordable;
- if it does not, then it must mean (under the W.A.R.P.) that $x(p, w)$ was not affordable;
- or else, the W.A.R.P. is violated.



Solution: does this demand satisfy W.A.R.P.?

- We have a useful result: if a demand $x(p, w)$ satisfies Walras' law, homogeneity of degree zero and the W.A.R.P. at **any** (p, w) , then the Slutsky $S(w, p)$ must be *negative semidefinite*.
- We checked that the matrix is negative semidefinite for $p = (1, 1, 1)$, but it is so for every (p, w) ?
- We must hence check the definiteness of the matrix for a generic p .
- Let's consider the price of the third commodity as a variable, ε : $p = (1, 1, \varepsilon)$.
- let's consider the 2×2 submatrix A obtained deleting last row and column:

$$A = \frac{1}{(2+\varepsilon)^2} \begin{bmatrix} -2-\varepsilon & 1+2\varepsilon \\ 0 & -3\varepsilon \end{bmatrix}$$

- If we choose the vector $z = [1, 3, 0]$ and we calculate the quadratic form for A , we get

$$z' A z = [1, 3] \frac{1}{(2+\varepsilon)^2} \begin{bmatrix} -2-\varepsilon & 1+2\varepsilon \\ 0 & -3\varepsilon \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \dots = \frac{1-20\varepsilon}{(2+\varepsilon)^2}$$

- which is positive for $\varepsilon > 0$ small (i.e. $\varepsilon < 0.05$).
- Hence $S(w, p)$ is not negative semidefinite for all price vectors, and W.A.R.P. is *not* satisfied.



2. MWG 2.F.17: demand

In an L -commodity world, a consumer's Walrasian demand function is

$$x_k(p, w) = \frac{w}{\sum_{l=1}^L p_l}, \text{ for } k = 1, \dots, L$$

1. Is this demand homogeneous of degree zero in (p, w) ?
2. Does it satisfy Walras' Law?
3. Does it satisfy the Weak Axiom?
4. Compute the Slutsky substitution matrix for this demand function. Is it negative semidefinite? Negative definite? Symmetric?



Solution: points 1 and 2

Homogeneity of degree zero

- Let's just apply the definition: $x_k(\alpha p, \alpha w) = x_k(p, w)$

$$x_k(\alpha p, \alpha w) = \frac{\alpha w}{\sum_{l=1}^L \alpha p_l} = \frac{\alpha w}{\alpha \sum_{l=1}^L p_l} = \frac{w}{\sum_{l=1}^L p_l} = x_k(p, w)$$

Walras' law

- Again, let's apply the definition: $p \cdot x(p, w) = w$

$$p \cdot x(p, w) = \sum_{l=1}^L p_l x_l(p, w) = \sum_{l=1}^L p_l \frac{w}{\sum_{l=1}^L p_l} = w$$



Solution: point 3

- Suppose the situation is a potential violation of W.A.R.P.:
 - $p' \cdot x(p, w) \leq w'$ and $p \cdot x(p', w') \leq w$;
 - This means that both bundles are affordable every time (but choices differ).
 - For the W.A.R.P. to hold, the bundles $x(p, w)$ and $x(p', w')$ must be *the same*

Proof.

1. By substitution, $p' \cdot x(p, w) \leq w'$ implies $\sum_I p'_I \frac{w}{\sum_I p_I} \leq w'$, i.e. $\frac{w}{\sum_I p_I} \leq \frac{w'}{\sum_I p'_I}$
2. and $p \cdot x(p', w') \leq w$ implies $\sum_I p_I \frac{w'}{\sum_I p'_I} \leq w$, i.e. $\frac{w'}{\sum_I p'_I} \leq \frac{w}{\sum_I p_I}$
3. Therefore it must be $\frac{w'}{\sum_I p'_I} = \frac{w}{\sum_I p_I}$;
4. Which means $x(p, w) = x(p', w')$





Solution: point 4

Slutsky substitution matrix

- Instead of computing element by element, we will compute the whole matrix in one go,
- since we will see it is more straightforward.
- The derivative of demand w.r.t. prices is given by the following $L \times L$ matrix:

$$D_p x(p, w) = -\frac{w}{(\sum_I p_I)^2} \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix}$$

- The compensated wealth effects are instead given by the column vector of pure wealth effects multiplied by the row vector obtained transposing the demand function $x(p, w)$ for all goods:

$$D_w x(p, w) \cdot x(p, w)^T = \frac{1}{\sum_I p_I} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \frac{w}{\sum_I p_I} \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}$$



...continued

- The two above expressions must be combined to get $S(w, p)$:

$$S(w, p) = -\frac{w}{(\sum_I p_I)^2} \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix} + \frac{w}{(\sum_I p_I)^2} \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

- ...which, being the null matrix, is symmetric;
- and is negative semidefinite, even if not negative definite,
- since it has all eigenvalues equal to zero.



3. MWG 3.B.2: monotonicity

The preference relation \lesssim defined on the consumption set $X = \mathbb{R}_+^L$ is said to be *weakly monotone* if and only if $x \geq y$ implies $x \lesssim y$. Show that if \lesssim is transitive, locally non satiated and weakly monotone, then it is monotone.



Recap: monotonicity, local non-satiation

Definition (Monotonicity)

- The preference relation \succsim on X is *monotone* if $x \in X$ and $y \succsim x$ implies $y \succ x$.
- Intuitively, monotonicity implies that all goods are desirable: if a new bundle has more of every good, it is strictly preferred to the old bundle.

Definition (Local non-satiation)

- The preference relation \succsim on X is *locally non-satiated* if for every $x \in X$ and every $\varepsilon > 0$ there is $y \in X$ such that $\|y - x\| \leq \varepsilon$ and $y \succ x$.
- Intuitively, under local non-satiation it is enough to move a very small step from x and there will be a y that is preferred to x .
- Local non-satiation does not impose all goods to be desirable: y can also have less of some goods (but not of all, unless they are all 'bads').



Solution: Proof I

We have to prove that:

$$\begin{array}{c} \text{transitive} \\ \succsim \\ \text{loc.non-sat.} \\ \text{weakly monotone} \end{array} \Rightarrow \begin{array}{c} \text{monotone } (x \gg y \Rightarrow x \succ y) \\ \succsim \end{array}$$

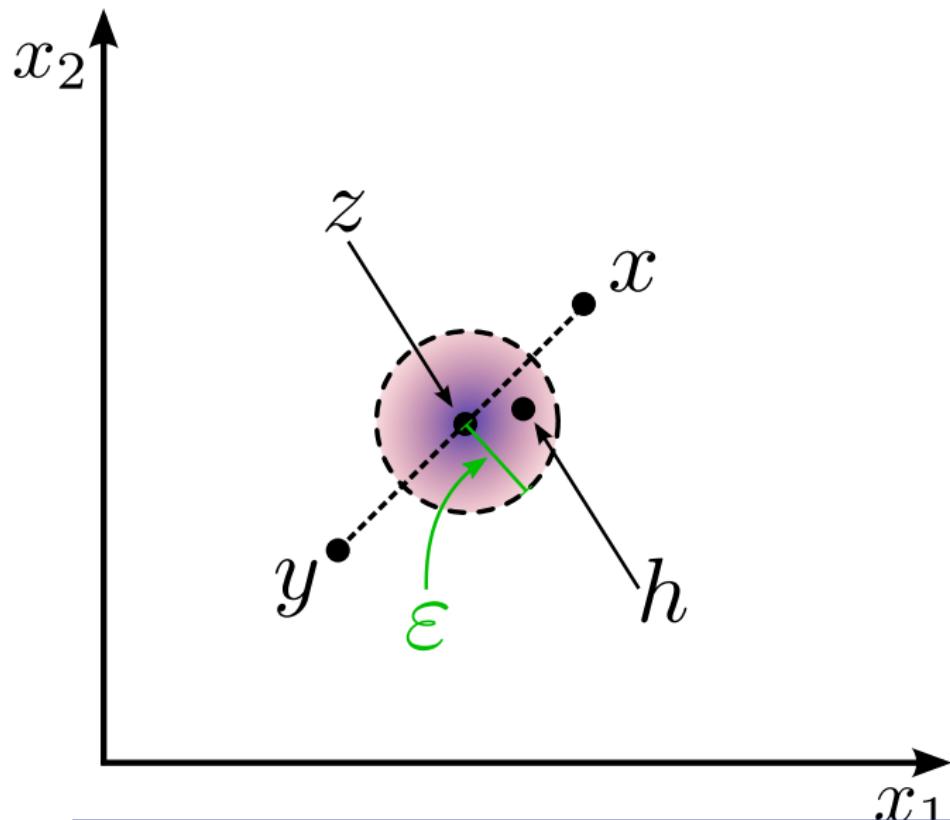
- we will be using weak monotonicity and local non-satiation

Monotonicity.

- Let's take two bundles x, y with $x \gg y$ (i.e. $x_i > y_i \forall i$).
- By weak monotonicity, $x \succsim y$.
- Let's take a linear combination $z = \alpha x + (1 - \alpha)y$, $\alpha \in (0, 1)$
- We know that it must be $x \gg z \gg y$;
- by weak monotonicity, $x \succsim y \succsim z$ (see figure)



Solution: Graphics





Solution: Proof II

Monotonicity.

- Using local non satiation, we know that:
- $\forall \varepsilon > 0 \exists h : \|z - h\| < \varepsilon$, such that $h \succ z$.
- ε can be chosen as small as we want, i.e. as small as to satisfy:
- $\varepsilon < \min\{\|x - z\|; \|z - y\|\}$.
- Choosing ε according to that condition ensures that $x \gg h \gg y$;
- By weak monotonicity, this means $x \gtrsim h \gtrsim y$.
- Summarising what we know:
- $x \gtrsim z \gtrsim y$, $x \gtrsim h \gtrsim y$ and $h \succ z$.
- putting it all together we must have
- $x \gtrsim h \succ z \gtrsim y$, which implies, by a theorem we proved in Problem Set 1 (have a look), $x \succ y$, Q.E.D.





Cobb-Douglas utility

Cobb-Douglas utility functions in \mathbb{R}^2 , for two goods x, y are given by the general formula

$$U = x^\alpha y^{1-\alpha}$$

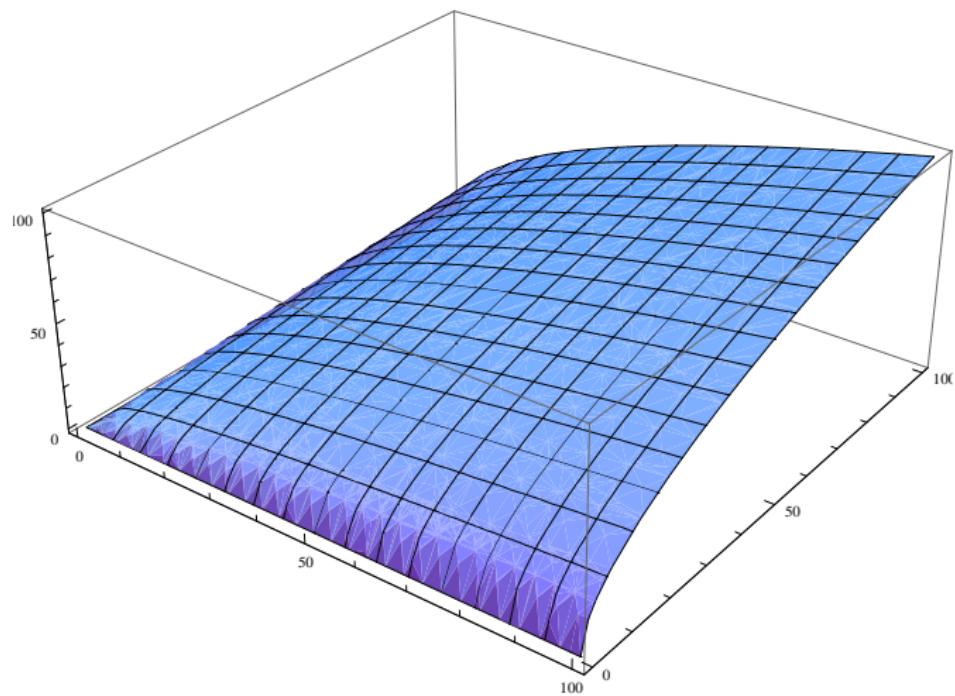
The function can be shown to be:

- monotonically increasing in the two arguments;
- concave;
- twice continuously differentiable with $U' > 0$ and $U'' < 0$;
- in other words, what economists call *well-behaved*.

We can have a look at both the 3D visualisation of the function and the level curves (indifference curves)

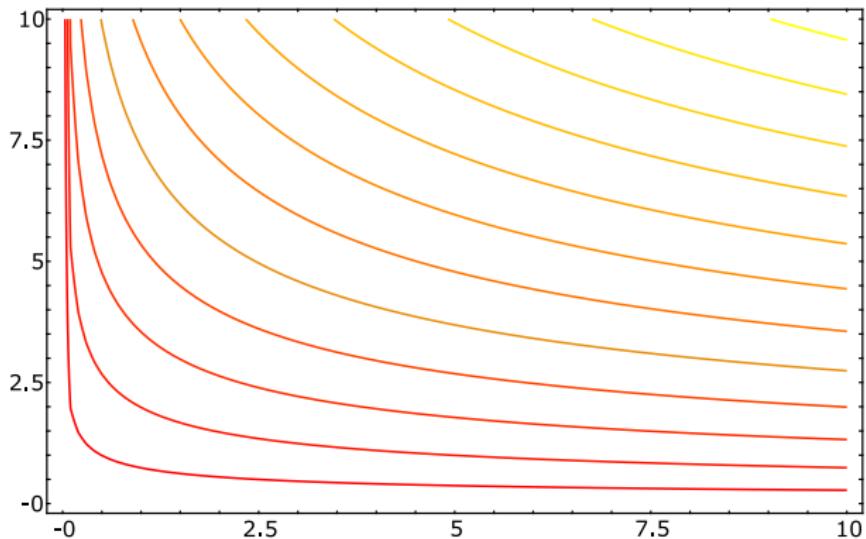


Cobb Douglas in 3D





Level (indifference) curves of Cobb-Douglas





Solving a maximisation problem: steps

1. Gather ideas on the set of alternatives X ;
2. Write down the budget constraint (and all other constraints);
3. Study the utility function; If more suitable, apply a strictly increasing (decreasing) transformation.
4. Write down the UMP as a Lagrangean (or as an unconstrained problem if you can);
5. Display and solve F.O.C.s;
6. The demand function is the set of values that maximises the utility for any given vector (p, w) :
7. Hence, by plugging the values obtained with the F.O.C. in the budget constraint, and solving, we get the demand functions $x_I(p, w)$, $\forall I = 1 \dots L$.



UMP with Cobb-Douglas utility I

Exercise

Let the utility function of a consumer be $U = x_1^\alpha x_2^{1-\alpha}$, and its budget constraint be $x_1 p_1 + x_2 p_2 \leq w$. Solve the Utility Maximisation Problem (UMP) and derive the demand functions $x_i(p, w)$, $i = 1, 2$ and the indirect utility function $v(p, w)$.

- Since U is increasing at all (p, w) , the budget constraint will always be satisfied with equality;
- We can then formalise the UMP as

$$\max x_1^\alpha x_2^{1-\alpha}, \text{ s.t. : } x_1 p_1 + x_2 p_2 = w$$

- The lagrangean can be setup to be:

$$\mathcal{L} = x_1^\alpha x_2^{1-\alpha} + \lambda(w - x_1 p_1 - x_2 p_2)$$



UMP with Cobb-Douglas utility II

- We can then compute the first order conditions (FOC), to be

$$\begin{cases} \frac{\partial U}{\partial x_1} = 0 \Rightarrow \alpha x_1^{\alpha-1} x_2^{1-\alpha} = \lambda p_1 \\ \frac{\partial U}{\partial x_2} = 0 \Rightarrow (1-\alpha) x_1^{\alpha} x_2^{-\alpha} = \lambda p_2 \end{cases}$$

- Dividing the first condition by the second we get

$$\frac{\alpha}{1-\alpha} \left(\frac{x_2}{x_1} \right) = \frac{p_1}{p_2}, \text{ i.e. } \Rightarrow p_1 x_1 = \frac{\alpha}{1-\alpha} p_2 x_2$$

- And by combining this condition with the budget constraint, and solving by x_1 and x_2 in turn, we get

$$x_1(p, w) = \frac{\alpha w}{p_1} \quad x_2(p, w) = \frac{(1-\alpha)w}{p_2}$$

- which you can verify to be homogeneous of degree zero and satisfying Walras' law.



UMP: indirect utility function

- The indirect utility function $v(p, w)$ is the *value function* of the maximisation problem
- It is hence defined as $v(p, w) = u(x(p, w))\dots$
- ...i.e. by plugging the walrasian demand $x(p, w)$ into the utility function $u(x)$.
- It gives the value of the maximum attainable utility given prices and wealth.
- In this case, $v(p, w)$ is given by

$$v(p, w) = u(x(p, w)) = \left(\frac{\alpha w}{p_1}\right)^\alpha \left(\frac{(1-\alpha)w}{p_2}\right)^{1-\alpha}$$