Problem Set II: budget set, convexity

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Exercises will be solved in class on *January 25th, 2010*

Recap: Walrasian Budget set, definition

Definition (Walrasian budget set)

A Walrasian budget set is given by $B_{p,w} = \{x \in X : \; p \cdot x \leq w\}$, when prices are p and wealth w.

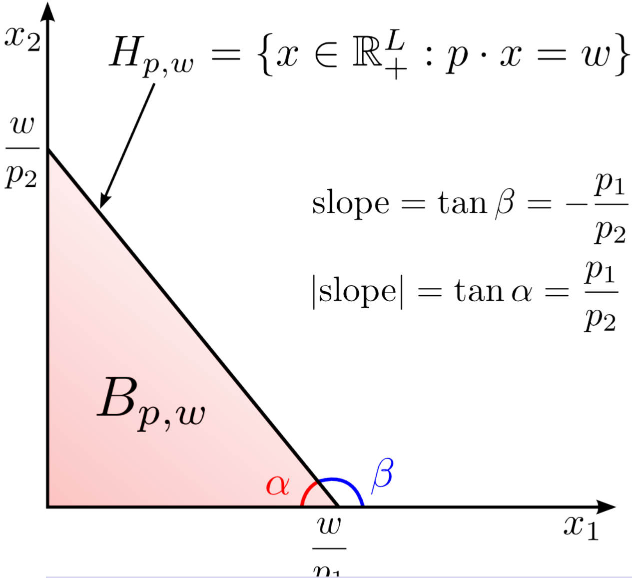
- Note that instead of wealth endowments ω can be used.
- It represents all consumption bundles that are affordable given prices and wealth.

Definition (Budget hyperplane)

A budget hyperplane is the upper contour of B: $H_{p,w} = \{x \in X : \; p \cdot x = w\}$.

- It represent all the consumption bundles that are just affordable (the consumer fully expends his wealth) given p and w.

Recap: Walrasian Budget set, graphics



Recap: Assumptions on demand, definitions

Definition (Homogeneity of degree zero)

$x(p, w)$ is homogeneous of degree zero if $x(\alpha p, \alpha w) = x(p, w), \forall p, w, \alpha > 0$.

- the consumer is not affected by 'money illusion'
- if prices and wealth both change in the same proportion, then the individuals consumption choice does not change.

Definition (Walras' law)

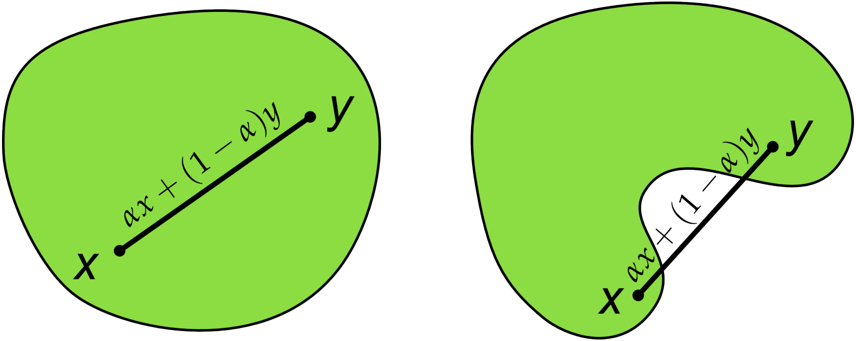
A demand function $x(p, w)$ satisfies Walras law (at the individual level) if $\forall p \gg 0,\ w > 0,\ p \cdot x = w, \forall x \in x(p, w)$.

- the consumer fully expends his wealth
- there is some good that is always desirable (not all goods are *bads*)
- savings are allowed once they are seen as just another commodity

Recap: convexity, sets

Definition (Convexity, sets)

A set S is convex if for all $x, y \in S$, then $\alpha x + (1-\alpha) y \in S$, $\forall \alpha \in (0,1)$.

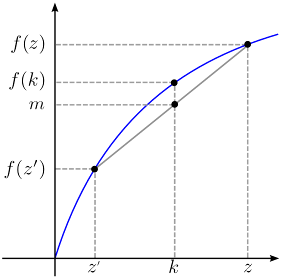


Recap: convexity, functions

Definition (Convexity, functions)

A function $f(x)$ defined on an interval I is convex if, for all $x, y \in I$ and for all $0 < \alpha < 1$:

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

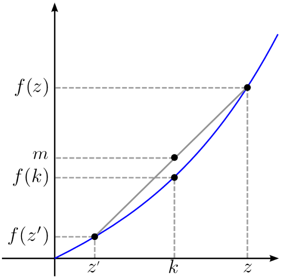


$$k = \alpha z + (1-\alpha) z'$$
$$m = \alpha f(z) + (1-\alpha) f(z')$$

Concavity: $f(k) \geq m$

Convexity: $f(k) \leq m$

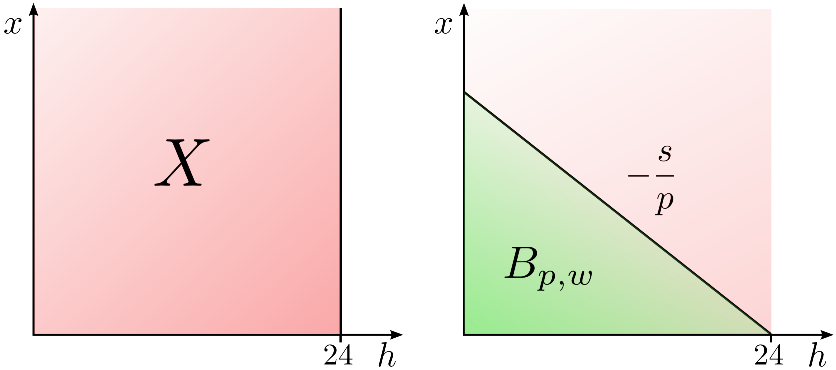
Linear: $f(k) = m$



1. MWG 2.D.2: building consumption and budget sets

A consumer consumes one consumption good x and hours of leisure h. The price of the consumption good is p, and the consumer can work at a wage rate of $s = 1$. What is the consumption set X? What is the consumer Walrasian Budget set? Write them down analytically and draw geometrically in $\mathbb{R}^2_+$.

Graphical Solution



Analytical solution

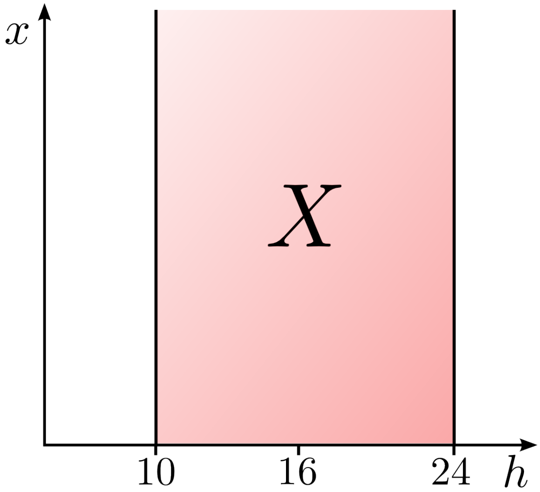
Analytical Solution

- Consumption set is $X = \{(x, h) \in \mathbb{R}^2_+ : h \leq 24\}$
- Budget set is $B_{p,w} = \{(x, h) \in \mathbb{R}^2_+ : px + sh \leq w\}$
- Since we know that $w = 24$ and $s = 1$, budget set boils down to $B_{p,w} = \{(x, h) \in \mathbb{R}^2_+ : px + h \leq 24\}$
- that defines a Budget line with equation $x = \dfrac{24 - h}{p}$, with slope $-\dfrac{1}{p}$

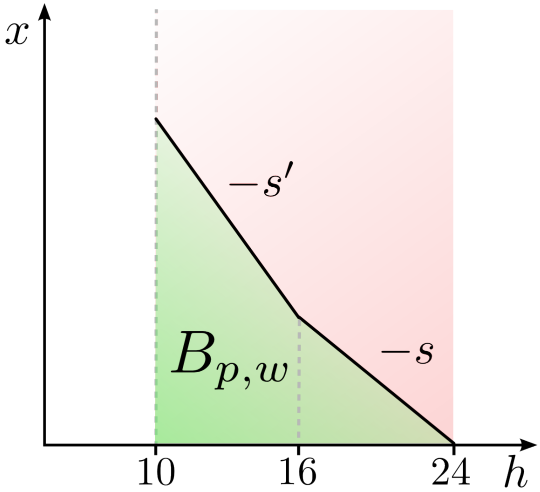
2. MWG 2.D.4 (with changes): convexity consumption and budget sets

A consumer consumes one consumption good x and hours of leisure h. The price of the consumption good is $p = 1$. The consumer can work at a wage rate of $s = 1$ for 8 hours, and at wage $s' > s$ for extra time; however, he can work only up to 14 hours a day.
Draw the budget set in $\mathbb{R}^2_+$ [*Hint: it's very similar to the one on MWG*] and derive an analytical expression for it; then show both graphically and analytically that the budget set you drew and derived is not convex.

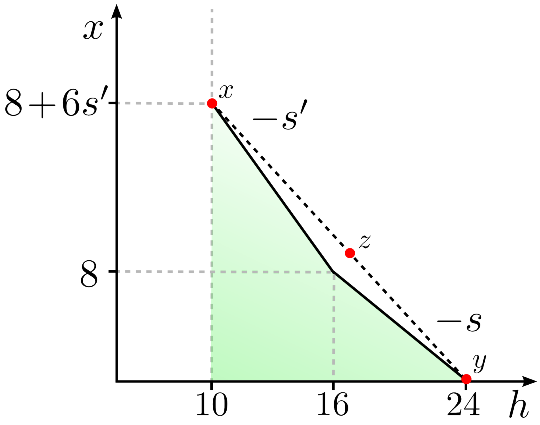
Graphical solution: X set



Graphical solution: budget set



Graphical solution: convexity



Analytical solution

X and budget sets

- Consumption set is $X = \{(x, h) \in \mathbb{R}^2_+ : 10 \leq h \leq 24\}$
- Wealth w is given by hours times salary, p and s are at 1;
- Budget set is
$$B_{p,w} = \begin{cases} \{(x, h) \in \mathbb{R}^2_+ : x + h \leq 24\} & \text{for } 16 \leq h \leq 24 \\ \{(x, h) \in \mathbb{R}^2_+ : x + s'h \leq 8 + 16s'\} & \text{for } 10 \leq h < 16 \end{cases}$$

Convexity

- To show that a set is not convex it is enough to find *one* exception to the rule.
- Definition of convexity: if $a, b \in S$, S is convex if and only if:
- $\alpha a + (1 - \alpha) b \in S$, $\forall \alpha \in (0, 1)$.
- Let's take two points that belong to B, namely $y\,(24, 0)$ and $x\,(10, 8 + 6s')$
- Now let's take the midpoint, i.e. a convex combination with $\alpha = 0.5$
- The midpoint z will have coordinates $(17, 4 + 3s')$.
- The utmost reachable point on the budget line has coordinates $(17, 7)$;
- since by hypothesis $s' > s$, $s' > 1$; hence $4 + 3s' > 7$ and $z \notin B$

3. MWG 2.E.1.

Suppose $L = 3$ and consider the demand function $x(p, w)$ defined by:

$$x_1(p, w) = \frac{p_2}{p_1 + p_2 + p_3} \frac{w}{p_1}$$

$$x_2(p, w) = \frac{p_3}{p_1 + p_2 + p_3} \frac{w}{p_2}$$

$$x_3(p, w) = \frac{\beta p_1}{p_1 + p_2 + p_3} \frac{w}{p_3}$$

Does this demand function satisfy homogeneity of degree zero and Walras' law when $\beta = 1$? What about when $\beta \in (0, 1)$?

Solution: Homogeneity of degree zero

- Homogeneity of degree zero means that multiplying all arguments of a function by a constant does not change the function.
- Formally, $f(\alpha a, \alpha b) = f(a, b),\ \forall \alpha$.
- So, we will do just that: multiply all arguments by α.

$$x_1(\alpha p, \alpha w) = \frac{\alpha p_2}{\alpha p_1 + \alpha p_2 + \alpha p_3} \frac{\alpha w}{\alpha p_1} = x_1(p, w)$$

- This is so as α simplifies everywhere.
- the very same calculation can be carried out for the other two cases, with similar results:
- Hence, $x(p, w)$ is homogeneous of degree zero.

Solution: Walras' law

- (individual level) Walras' law states that the consumer fully spends his wealth:
- $x(p, w)$ satisfies the law if $\forall p \gg 0$, $\forall w > 0$, $p \cdot x = w$, $\forall x \in x(p, w)$.
- To check, it is then enough to apply the definition, carrying on the vector product of p and a generic x.

- We will hence calculate $p \times x(p, w) = p_1 x_1(p, w) + p_2 x_2(p, w) + p_3 x_3(p, w)$
- Calculations show this to be equal to

$$p \cdot x(p, w) = \frac{\beta p_1 + p_2 + p_3}{p_1 + p_2 + p_3} w$$

- Which is equal to w if $\beta = 1$, but it is NOT otherwise.
- Hence $x(p, w)$ satisfies Walras' law if and only if $\beta = 1$.

Added Magic. MWG 2.E.4: demand, Engel functions

Show that if $x(p, w)$ is homogeneous of degree one with respect to w, i.e. $x(p, \alpha w) = \alpha x(p, w)$ for all $\alpha > 0$, and satisfies Walras' law, then $\varepsilon_{lw}(p, w) = 1$ for every l. Interpret. Can you say something about $D_w x(p, w)$ and the form of the Engel functions and curves in this case?

Solution: elasticity equals 1, I

- Start with the definition of elasticity:

$$\varepsilon_{lw}(p, w) = \frac{\partial x_l(p, w)}{\partial w} \frac{w}{x_l(p, w)}, \text{ for all } l$$

- in which the first term is the derivative of demand w.r.t. wealth w for good l
- i.e., for all goods, it is $D_w x(p, w)$.
- Where can we get that from?
- Let's differentiate by α the definition $x(p, \alpha w) = \alpha x(p, w)$
- By the chain rule, we know that:

$$\frac{\partial x(p, \alpha w)}{\partial \alpha} = \frac{\partial x(p, \alpha w)}{\partial \alpha w} \frac{\partial (\alpha w)}{\partial \alpha}$$

- Since $x(p, \alpha w) = \alpha x(p, w)$, the lhs of the equation becomes

$$\frac{\partial x(p, \alpha w)}{\partial \alpha} = \frac{\partial \alpha x(p, w)}{\partial \alpha} = x(p, w)$$

- and the rhs becomes

$$\frac{\partial x(p, \alpha w)}{\partial \alpha w} \frac{\partial (\alpha w)}{\partial \alpha} = D_{\alpha w} x(p, \alpha w) w$$

- hence, summing up

Solution: elasticity equals 1, II

- Evaluating at $\alpha = 1$ we get

$$D_w x(p, w) = \frac{x(p, w)}{w} \text{ for every good.}$$

- hence, for the good l we have

$$\frac{\partial x_l(p, w)}{\partial w} = \frac{x_l(p, w)}{w}$$

- which is what we were looking for.
- By plugging this result into the definition of elasticity, we get

$$\varepsilon_{lw}(p, w) = \frac{\partial x_l(p, w)}{\partial w} \frac{w}{x_l(p, w)} = \frac{x_l(p, w)}{w} \frac{w}{x_l(p, w)} = 1$$

- as requested by the exercise.

Considerations

- The result above tells us that an increase in income will increase the consumption of all goods by the same amount.
- Using the homogeneity assumption we can deduce that $x(p, w)$ is linear in w.
- Then, $\dfrac{x(p, w)}{w}$ defines a function of p only, $x(p, 1)$.
- Then, the matrix of wealth effects $D_w x(p, w)$ is a function of p only.

- The wealth expansion path is the locus of demanded bundles for a given set of prices when we let the wealth vary: $E_p = \{x(p, w) : w > 0\}$
- Since linear wealth elasticity implies that wealth effects are only functions of prices,
- wealth just increases all quantities demanded proportionally;
- hence the wealth expansion paths are straight lines, *rays* through $x(p, 1)$
- this defines *homothetic* preferences.

Homotheticity, graphics