Games and Strategy TA 7
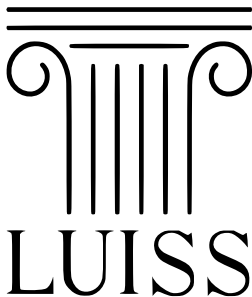*Core*

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October 28, 2010

Studying coalitions

Coalitional games study the behaviour of groups of players, or *coalitions*. Players form groups, and each group performs some action. A coalitional game with transferable utility is given by:

- a finite set N of players
- a function v that associates a real number $v(S)$ to any $S \subseteq N$.

We further assume:

1. That the actions of players outside of each S do not affect the total payoff to be distributed to the members of S;

2. That the game is *cohesive*, i.e. that it is optimal for the grand coalition N to form:
$$v(N) \geq \sum_{i=1}^{N} v(S_i), \text{ for every partition } \{S_1, \ldots, S_i\} \text{ of } N$$

Any such game can be denoted $\langle N, v(S) \rangle$

- The core, as the Nash equilibrium, is built on top of the *no-deviation* property
- But it considers deviations by coalitions, not by single players.

Core: definition

The core is the *set* of actions a_N of the grand coalition N such that no coalition $S \subseteq N$ has an action a_S that all its members prefer to a_N.

In the core, no coalition S can *improve upon* a_N.

When game is T.U., we have formally that an action a_N of the grand coalition is in the core iff

$$\sum_{i \in S} x_i(a_N) \geq v(S), \text{ for all } S \subseteq N$$

so that no player has an incentive to create, alone or joint with other players, any coalition S.

Game setup

- $N = 3$ players
- For every i, preferences are given by the payments, $x_i = (x_1, x_2, x_3)$
- $v(S)$ is such that

$$v(N) = 1$$
$$v(\{i,j\}) = 1$$
$$v(\{i\}) = 0$$
$$v(\varnothing) = 0$$

- Game is cohesive, has transferable utility;
- What is the core?

Exercise: majority voting

Solution

For an action a_N to be in the core, it must be such that no coalition can deviate and secure a higher payoff to all of its members; i.e.

$$v(S) \leq \sum_{i \in S} x_i(a_N)$$

In this case, we have the following set of inequalities:

$$\begin{cases} 1 \leq x_1 + x_2 \\ 1 \leq x_2 + x_3 \\ 1 \leq x_1 + x_3 \\ x_1 + x_2 + x_3 = 1 \\ 0 \leq x_i, \forall i \end{cases}$$

Solution: empty core

The system boils down to

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 + x_3 \geq \dfrac{3}{2} \end{cases}$$

Hence the core of this game is empty.

Exercise: majority voting with a dominant palyer

Same game as before, but now palyer 1 has three votes.

Game setup

- $N = 3$ players
- For every i, preferences are given by the payments, $x_i = (x_1, x_2, x_3)$
- Player 1 has three votes: no coalition without her can have the majority. Hence $v(S)$ is:

$$\begin{cases} v(S) = 1 & \text{if } 1 \in S \\ v(S) = 0 & \text{if } 1 \notin S \end{cases}$$

- Note that in this case player one is a *dictator*: $v(S) = 1$ *iff* $1 \in S$

Core of the game is $a_N = (1, 0, 0)$

Three legislators have to pass bills by majority voting. Once a majority is formed, it has to choose which policies to implement out of the set $\{A, B, C\}$. Payoffs are given by

	A	B	C
1	2	-1	1
2	1	2	-1
3	-1	1	2

What actions are in the core?

Solution strategy

We can check each action of the grand coalition N in turn, and see if any subcoalition can improve on it.

> The only action in the core is $a_N = (A, B, C)$, i.e. all policies pass.

Same as before, but now the payoffs are given by

	A	B	C
1	2	-1	-1
2	-1	2	-1
3	-1	-1	2

Solution

The core is empty.

Exercise: landowner and workers, I

Landless workers

All land in a country belongs to a landowner. When k people work the land, the output is $f(k)$, f increasing at a decreasing rate (f concave: this means $f(k) - f(k-1)$ is decreasing in k, or else $f'(k) > 0$ and $f''(k) < 0$).

What action a_N of the grand coalition is in the core?

Solution

The action $a_N = (f(1), 0, \ldots, 0)$ is in the core; All the actions in which each worker is paid up to her marginal product are in the core, i.e. all actions for which $x_j = f(k) - f(k-1)$ and player i (the owner) gets the rest.

Unionized workers

All land in a country belongs to a landowner. When k people work the land, the output is $f(k)$, f increasing at a decreasing rate (f concave: this means $f(k) - f(k-1)$ is decreasing in k, or else $f'(k) > 0$ and $f''(k) < 0$). The workers act collectively in a trade union - i.e. act as one man instead of being in competition between each other. This translates into the fact that any group of workers not including every worker refuses to work with the landowner. What action a_N of the grand coalition is in the core?

Solution

The action $a_N = (f(1), 0, \ldots, 0)$ is in the core; All other feasible allocations of the amount $f(k)$ are in the core.